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Isothermal Hydrodynamics of Orthorhombic Nematics

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Differential equations governing elastic and flow behaviour of compressible orthorhombic biaxial nematics are derived using the Ericksen-Leslie approach for uniaxial nematics. The preferred direction of orientation is represented by a mutually orthogonal triad of unit vectors (directors), each of which is a diad axis of symmetry. The expression for non-dissipative stress is deduced as a consequence of entropy inequality; the free energy density is found to satisfy a symmetry relation similar to that for uniaxial nematics. The expression for viscous stress derived on general symmetry assumptions depends on 21 viscosity coefficients. Use of the dissipative function approach reduces this number to 15 via three Onsager relations and three symmetry relations; the viscous stress becomes identical to that derived by Saupe.[2] Under uniaxial symmetry about one director, the expression for the viscous stress reduces to that for a compressible uniaxial nematic. It seems possible to determine combinations of some of the viscosity coefficients through the determination of apparent viscosity in shear flow and plane Poiseuille flow. It also seems reasonable to expect transverse flow effects in plane Poiseuille flow, similar to those observed for uniaxial nematics.

1. INTRODUCTION

Since the discovery of the biaxial nematic phase in a multicomponent system by Yu and Saupe,[1] a number of theories have been proposed[2–10] for describing the hydrodynamic behavior of these materials. Out of these theories, the one given by Saupe[2] for a compressible biaxial with local orthorhombic symmetry appears to come closest in approach to the Ericksen-Leslie theory of uniaxial nematics.[11–16] Some of the other theories[3–10] do bear some similarities to one another but are not discussed here. Saupe starts from the elastic theory of crystals[17] and generalises it by introducing degrees of freedom associated with orientational order characterised by a mutually orthonormal triad of vectors **a**, **b**, **c**, each of which is a two-fold axis of symmetry. He shows from symmetry considerations that the free energy of deformation is determined by a bulk modulus and 15 curvature elastic constants, of which 3 contribute to surface terms. This conclusion has also been arrived at in ref. 10. There are three more elastic constants which describe spontaneous deformation if it is present. Starting from the dissipative function which is written as a quadratic form in velocity gradient and velocity of rotation of the director field relative to the fluid, Saupe further shows that the dissipative stress σ'_{ij} is described by 15 viscosity coefficients, of which 6 describe viscous torque on the director field. He suggests experiments involving uniform shear and magnetic alignment which could help determine combinations of some of the viscosity coefficients which characterise incompressible flow. He also studies flow alignment and its stability in certain simple configurations.

In this communication a theory describing isothermal hydrodynamics of biaxial nematics is derived following the Ericksen–Leslie approach. Symmetry considerations lead to an expression for σ'_{ji} which reduces to that for uniaxial nematics. Determination of apparent viscosity in simple shear and plane Poiseuille flows appears to provide a way of formally comparing the present approach with the dissipative function approach. Plane Poiseuille flow seems to have transverse flow and pressure effects which are reminiscent of those for uniaxial nematics.

2. CONSERVATION LAWS

The approach adopted in this work closely follows the one given for uniaxial nematics in ref. 15 and is similar to that of ref. 13 and 14 with some differences.

We consider a compressible orthorhombic biaxial nematic liquid crystal at each point x_k of which the preferred direction of orientation is represented by an orthonormal triad of dimensionless unit vectors **a**, **b**, **c** whose magnitude is assumed to be fixed. Each of **a**, **b**, **c** is a diad axis of symmetry. Temperature T is assumed to be constant.

At time t the conservation laws of mass, linear momentum, energy and angular momentum as also the biaxial analogues of Oseen's equations for a volume V of the fluid bounded by surface A can be written as,

$$\frac{d}{dt}\int_V \rho\, dV = 0 \tag{1}$$

$$\frac{d}{dt}\int_V \rho v_i\, dV = \int_V \rho f_i\, dV + \oint_A \sigma_{ji}\, dA_j \tag{2}$$

$$\frac{d}{dt}\int_V \left[\frac{1}{2}\rho v_i v_i + \rho U + \frac{1}{2}\sum_a \rho\rho_a \dot{a}_i \dot{a}_i\right] dV = \int_V \left[\rho f_i v_i + \rho \sum_a G_i^a \dot{a}_i\right] dV + \oint_A \left[\sigma_i v_i + \sum_a s_i^a \dot{a}_i\right] dA \tag{3}$$

$$\frac{d}{dt}\int_V e_{ijk}\left[\rho x_j v_k + \sum_a \rho\rho_a a_j \dot{a}_k\right] dV = \int_V e_{ijk}\left[\rho x_j f_k + \rho \sum_a a_j G_k^a\right] dV + \oint_A e_{ijk}\left[x_j \sigma_k + \sum_a a_j s_k^a\right] dA \tag{4}$$

$$\frac{d}{dt}\int_V \rho_a \dot{a}_i\, dV = \int_V (\rho G_i^a + g_i^a)\, dV + \oint_A \pi_{ji}^a\, dA_j \text{ (CP)} \tag{5}$$

where ρ is the density, d/dt the material derivative, v_i the velocity, f_i the external body force per unit mass, σ_{ji} the stress tensor, U the internal energy per unit mass, $\sigma_i = \sigma_{ji}\nu_j$ the surface force per unit area across a plane whose outward drawn unit normal is ν_j, ρ_a = constant, the moment of inertia per unit mass, G_i^a the external director body force, g_i^a the intrinsic director body force, π_{ji}^a the director surface stress, $s_i^a = \pi_{ji}^a \nu_j$ the director surface force per unit area associated with **a** and $\dot{a}_i = da_i/dt$ (CP) CP refers to cyclic permutation over symbols a, b, c and A, B, C. As will become clear later, all nine of Eqs. (5) are not independent.

With Reynold's transport theorem and Gauss theorem (see for instance ref. 18), Eqs. (1)–(5) can be written as follows:

$$\dot{\rho} + \rho v_{k,k} = 0 \tag{6}$$

$$\rho \dot{v}_i = \sigma_{ji,j} + \rho f_i \tag{7}$$

$$\rho_a \rho \ddot{a}_i = \pi_{ji,j}^a + g_i^a + \rho G_i^a \text{ (CP)} \tag{8}$$

$$\rho \dot{U} = \sigma_{ji} d_{ij} + \sum_a \left[\pi_{ji}^a N_{ij}^a - g_i^a N_i^a\right] \tag{9}$$

$$\sigma_{jk} + \sum_a \left[\pi_{lk}^a a_{j,l} - a_j g_k^a\right] = \sigma_{kj} + \sum_a \left[\pi_{lj}^a a_{k,l} - a_k g_j^a\right] \tag{10}$$

where $$d_{ij} = (v_{i,j} + v_{j,i})/2, \omega_{ij} = (v_{i,j} - v_{j,i})/2 \tag{11}$$

$$N_i^a = \dot{a}_i - \omega_{ik} a_k, N_{ij}^a = \dot{a}_{i,j} - \omega_{ik} a_{k,j} \text{ (CP)} \tag{12}$$

Eq. (9) is obtained from Eq. (3) by using Eq. (10). N_i^a represents the velocity of rotation of director **a** relative to the fluid (CP). Since **a**, **b**, **c**, remain mutually orthonormal, N_i^a (CP) are not independent as is shown in Appendix I. It is however sufficient to use the relation describing their dependence for writing the final expression for σ'_{ji}. With S the entropy per unit mass and $F = U - TS$ the Helmholtz free energy per unit mass, Eq. (9) can be recast to express the rate of entropy generation and hence the entropy generation inequality:

$$\rho T\dot{S} = \sigma_{ji} d_{ij} + \sum_a \left[\pi_{ji}^a N_{ij}^a - g_i^a N_i^a\right] - \rho\dot{F} \geq 0 \tag{13}$$

3. CONSTITUTIVE ASSUMPTIONS AND ENTROPY INEQUALITY

Adopting the principle of equipresence[13,19] we assume that F, σ_{ji}, π_{ji}^a, g_i^a (CP) are single valued functions of the variables ρ, $v_{i,j}$, a_i, $a_{i,k}$, $\dot{a}_i$ (CP). It is possible to rewrite the list of independent variables by using Eqs. (11) and (12). However by the principle of material frame indifference[13,19] material properties are indifferent to the frame of reference or the observer. In order that this principle may be satisfied, the dependent variables have to be expressed as tensor functions of the independent variables which should themselves transform as tensors under general, time dependent changes of frame. This implies replacing $\dot{a}_i$ by N_i^a (CP) and $v_{i,j}$ by d_{ij}; ω_{ij} is excluded from the list of variables[13] as it can be varied arbitrarily by superposed rigid body rotations. Thus the independent variables are

$$\rho, d_{ij}, a_i, a_{i,k}, N_i^a \text{ (CP)} \tag{14}$$

Eqs. (1)–(5) show that external body forces f_i, G_i^a (CP) can be varied arbitrarily. Hence there is no restriction on the choice of the quantities (14) or their material and spatial derivatives. Using the chain rule of differential calculus to expand $\dot{F}$, the inequality (13) can be written in the form

$$\rho T\dot{S} = d_{ij}\left[\sigma_{ji} + \delta_{ij}\rho^2\, \partial F/\partial\rho + \rho\sum_a a_{k,i}\, \partial F/\partial a_{k,j}\right]$$
$$+\sum_a N_{ij}^a\left[\pi_{ji}^a - \partial F/\partial a_{i,j}\right] - \sum_a N_i^a\left[g_i^a + \partial F/\partial a_i\right]$$
$$+\omega_{ki}\sum_a\left[\rho a_k\, \partial F/\partial a_i + \rho a_{k,j}\, \partial F/\partial a_{i,j} + \rho a_{j,k}\, \partial F/\partial a_{j,i}\right]$$
$$-\rho\sum_a \dot{N}_i^a\, \partial F/\partial N_i^a - \rho\dot{d}_{ij}\, \partial F/\partial d_{ij} \geq 0 \tag{15}$$

Since $\dot{d}_{ij}$, $\dot{N}_i^a$ (CP) can be chosen arbitrarily

$$\partial F/\partial d_{ij} = 0; \qquad \partial F/\partial N_i^a = 0 \text{ (CP)} \tag{16}$$

Thus F is a function of ρ, a_i, $a_{i,j}$ (CP). Since the vorticity tensor ω_{ij} can be varied arbitrarily by superposed rigid rotations, it should not appear in (16). Hence the coefficient of ω_{ij} has to be symmetric and F should satisfy the condition

$$\sum_a\left[a_k\, \partial F/\partial a_i + a_{k,j}\, \partial F/\partial a_{i,j} + a_{j,k}\, \partial F/\partial a_{j,i}\right]$$
$$= \sum_a\left[a_i\, \partial F/\partial a_k + a_{i,j}\, \partial F/\partial a_{k,j} + a_{j,i}\, \partial F/\partial a_{j,k}\right] \tag{17}$$

The inequality (15) now becomes

$$\rho T\dot{S} = d_{ij}\left[\sigma_{ji} + \rho^2\delta_{ij}\, \partial F/\partial\rho + \rho\sum_a a_{k,i}\, \partial F/\partial a_{k,j}\right]$$
$$+\sum_a N_{ij}^a\left[\pi_{ji}^a - \partial F/\partial a_{i,j}\right] - \sum_a N_i^a\left[g_i^a + \partial F/\partial a_i\right] \geq 0 \tag{18}$$

Let the stress and intrinsic body force be written in the form

$$\sigma_{ji} = \sigma_{ji}^0 + \sigma_{ji}', \; g_i^a = g_i^{a0} + g_i^{a\prime} \text{ (CP)} \tag{19}$$

where the superscript 0 denotes the isothermal static deformation value and the prime the non-equilibrium or dissipative part. We desist from writing a similar expression for π_{ji}^a (CP) which can be shown to have only an equilibrium part.

From Eqs. (18) and (19)

$$\sigma^0_{ji} = -\rho^2\delta_{ij}\,\partial F/\partial\rho - \rho\sum_a a_{k,i}\,\partial F/\partial a_{k,j}$$

$$g_i^{a0} = -\rho\,\partial F/\partial a_i\,(\mathrm{CP}),\ \pi_{ji}^a = \pi_{ji}^{a0} = \rho\,\partial F/\partial a_{i,j}\,(\mathrm{CP}) \qquad (20)$$

The expression for equilibrium stress derived as a consequence of the entropy generation inequality is similar to that of ref. 2. Inequality (18) now becomes

$$\rho T\dot{S} = \sigma'_{ji}d_{ij} - \sum_a g_i^{a\prime}N_i^a \geq 0 \qquad (21)$$

Using Eqs. (12), (17), and (19) one gets

$$\sigma'_{ki} - \sigma'_{ik} = \sum_a \left[g_i^{a\prime}a_k - g_k^{a\prime}a_i \right] \qquad (22)$$

which expresses the viscous torque exerted on the director field. Substituting for $g_i^{a\prime}$ from Eq. (8), one can write the balance between the viscous torque, the elastic torque (determined by F) and the director angular acceleration

$$\sigma'_{ki} - \sigma'_{ik} = \sum_a \Big[a_k\left\{ \rho\rho_a\ddot{a}_i - \pi^a_{ji,j} - g_i^{a0} - \rho G_i^a \right\}$$
$$- a_i\left\{ \rho\rho_a\ddot{a}_k - \pi^a_{jk,j} - g_k^{a0} - \rho G_k^a \right\}\Big] \qquad (23)$$

Eq. (23) *clearly indicates that out of the nine equations* (8), *only three are independent*. These three, along with the force equation (7) should help determine three components of velocity and three quantities (say angles) which uniquely specify the director field **a**, **b**, **c**. Also π_{ji}^a, g_i^a (CP) are not uniquely specified since the transformations

$$g_i^{a0} \to g_i^{a0} + \gamma_a a_i - \beta_j^a a_{i,j}\,(\mathrm{CP})$$
$$\pi_{ji}^a \to \pi_{ji}^a + \beta_j^a a_i\,(\mathrm{CP})$$
$$\gamma_a = \text{constant}, \qquad \beta_j^a = \text{constant}\,(\mathrm{CP}) \qquad (24)$$

leave Eq. (23) unaffected.

4. FREE ENERGY AND DISSIPATIVE PART OF THE STRESS TENSOR

The fluid under study is an orthorhombic biaxial nematic. Each of **a**, **b**, **c** is a diad axis of symmetry. Motivated by the theory for uniaxial nematics,[15] we assume the following transformations which leave Eqs. (6)–(8) and the inequality (21) unaltered:

$$v_i \to v_i, \sigma_{ji} \to \sigma_{ji}, F \to F, \pi_{ji}^a \to \pi_{ji}^a, \pi_{ji}^b \to -\pi_{ji}^b, \pi_{ji}^c \to -\pi_{ji}^c$$
$$g_i^a \to g_i^a, g_i^b \to -g_i^b, g_i^c \to -g_i^c, G_i^a \to G_i^a, G_i^b \to -G_i^b,$$
$$G_i^c \to -G_i^c \text{ (CP)}$$

if
$$a_i \to a_i, b_i \to -b_i, c_i \to -c_i \text{ (CP)} \tag{25}$$

In addition we assume that the constitutive equations are invariant under reflections through planes containing any two of the (local) director axes. This excludes biaxial liquid crystals having spontaneous deformation. Following Saupe[2] (whose approach is similar to that of Frank[20]) one finds that

$$\begin{aligned}\rho F = F_0 + \frac{1}{2}\lambda(u_{ii})^2 + \frac{1}{2}\sum_a \Big[& k_{aa}(a_i b_{k,i} c_k)^2 + k_{ab}(a_i a_{k,i} b_k)^2 \\ & + k_{ac}(a_i a_{k,i} c_k)^2 + 2c_{ab}(a_i a_{k,i} b_j b_{k,j}) \\ & + 2k_{0a}(a_i a_{k,i} - a_k a_{i,i})_{,k}\Big]\end{aligned} \tag{26}$$

where λ (dyne cm^{-2}) is the bulk modulus, u_{ii} the compressional strain, k_{aa}, k_{ab}, k_{ac}, c_{ab}, k_{0a} (CP, dyne) curvature elastic constants of which k_{0a} contribute to surface terms. Positive definiteness of F implies that

$$\begin{gathered} k_{aa} \geq 0, k_{bc} \geq 0, k_{ac} \geq 0, k_{aa}k_{bb} - 4k_{0c}^2 \geq 0, \\ k_{bc}k_{ac} - (2k_{0c} - c_{ab})^2 \geq 0 \text{ (CP)}, \\ k_{aa}k_{bb}k_{cc} - 16k_{0a}k_{0b}k_{0c} - 4\sum_a k_{aa}k_{ca}^2 \geq 0, \lambda \geq 0 \end{gathered} \tag{27}$$

F identically satisfies Eq. (17) which was derived on the basis of the positive definiteness of entropy generation and on the independence of entropy generation on rigid body rotations of the entire system. Expressions for σ_{ji}^0, π_{ji}^a, g_i^{a0} (CP) calculated from Eqs. (20) and (26) satisfy Eq. (25). It has been shown by Saupe[2] that Eq. (26) reduces to

Frank's expression of the free energy of a uniaxial nematic[20] when uniaxial symmetry is assumed.

It is possible to arrive at Eq. (26) in a number of different ways. One can expand F tensorially up to second order in director gradients and elastic strain u_{ij} as

$$F = F_0 + A^{(1)}_{ijkl}u_{ij}u_{kl} + \sum_a A^{(a)}_{ijkl}a_{i,j}a_{k,l} + \sum_a A^{(ab)}_{ijkl}a_{i,j}b_{k,l} + \sum_a B^{(a)}_{ijkl}a_{i,j}u_{kl} + \sum_a C^{(a)}_{ij}a_{i,j}$$

where the different tensors A, B, C are expanded in terms of δ_{ij}, a_k, b_l, c_m and satisfy orthorhombic symmetry as per Eq. (25). All the scalars formed from the director vectors and their gradients can be expressed in terms of the 15 quantities in Eq. (26).

A simpler way of writing the curvature elastic part of Eq. (26) is to construct all scalars using $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and their gradients such that each scalar contains the gradients to first power. There are 27 such quantities out of which 9 are identically zero since $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors. Out of the remaining 18, nine are independent since the director vectors are mutually perpendicular. Using these 9 scalars such as $(a_i b_{k,i} c_k)$, we can form 45 products. Each product contains director gradients to second power. F can now be written as a linear superposition of these quantities. It is clear that if the material is triclinic, all the 45 terms are admissible and such a material would have 45 elastic constants.[21] If the material has monoclinic symmetry, only 25 out of the 45 terms are admissible; such a material would be described by 25 elastic constants.[21] For an orthorhombic material only 15 terms remain and these can be easily recast into the form given in Eq. (26).

The derivation of surface terms can be done on the lines suggested by Ericksen[22] for a uniaxial nematic. A general surface term can be written in the form $[V_i W_{k,i} - X_k Y_{i,i}]_{,k}$ where V, W, X, Y are chosen from (a, b, c). Out of the 45 independent terms, only six are admissible as they do not contain second derivatives. Thus a triclinic has 6 surface terms while a monoclinic nematic has 4 admissible terms.[21] For orthorhombic symmetry one is left with the three surface terms which appear in Eq. (26).

As assumed earlier, dissipation due to time rate of change of director gradients is ignored. With this in mind we assume that σ'_{ji} and $g_i^{a\prime}$ (CP) are functions of d_{ij}, a_k and N_i^a (CP) and ignore dependence on $a_{i,k}$ (CP). We further restrict dependence on the dissipative forces d_{ij} and N_i^a to first order. Following Ericksen[11] we write expansions of the form

$$\sigma'_{ji} = A^{(0)}_{ji} + \sum_a A^{(a)}_{jik} N^a_k + A^{(1)}_{jikm}\, d_{km}$$
$$g^{a\prime}_i = B^{(a0)}_i + \sum_a B^{(a)}_{ij} N^a_j + B^a_{ijk}\, d_{jk}\ (\mathrm{CP}) \tag{28}$$

where the tensors A, B, C satisfy certain symmetries as per Eq. (25) and are functions of a_i (CP). These tensors are expanded in the basis δ_{ij} and a_i (CP). At this stage it is convenient to introduce **N** which represents rotation of the director fields **a, b, c** relative to the fluid. As shown in Appendix I,

$$N_i = \sum_a \left(N^b_k c_k\right) a_i \quad \text{or} \quad N_i = \Omega^T_{kl} \sum_a b_l c_k a_i \tag{29}$$

where $\Omega^T_{kl} = -\Omega^T_{lk}$ represents the rotation. From Eqs. (25), (28) and (29) one finds

$$\begin{aligned}
\sigma'_{ji} = {} & \delta_{ij}[\alpha_1 A_a + \alpha_2 B_b + \alpha_3 C_c] + a_i a_j[\alpha_4 B_b + \alpha_5 C_c] \\
& + b_i b_j[\alpha_6 A_a + \alpha_7 C_c] + c_i c_j[\alpha_8 A_a + \alpha_9 B_b] \\
& + a_j b_i[\alpha_{10} N_k c_k + \alpha_{11} A_b] \\
& + a_i b_j[\alpha_{12} N_k c_k + \alpha_{13} A_b] + b_j c_i[\alpha_{14} N_k a_k + \alpha_{15} B_c] \\
& + b_i c_j[\alpha_{16} N_k a_k + \alpha_{17} B_c] \\
& + c_j a_i[\alpha_{18} N_k b_k + \alpha_{19} A_c] + c_i a_j[\alpha_{20} N_k b_k + \alpha_{21} A_c], \\
g^{a\prime}_i = {} & \lambda^{(a)}_1 N_k b_k b_i + \lambda^{(a)}_2 (b_i A_b - c_i A_c) + \lambda^{(a)}_3 N_k b_k c_i \\
& + \lambda^{(a)}_4\, d_{ik} a_k\ (\mathrm{CP})
\end{aligned} \tag{30}$$

where α_i and $\lambda^{(a)}_i$ are material constants having dimensions of viscosity and by Eq. (14) are functions of ρ; $A_a = a_r d_{rs} a_s$, $A_b = B_a = a_r d_{rs} b_s$ (CP). A term $\alpha_{22}\delta_{ij} + \alpha_{23} a_i a_j + \alpha_{24} b_i b_j + \alpha_{25} c_i c_j$ contributed by $A^{(0)}_{ji}$ is left out in anticipation of the positive definiteness of entropy generation.[23] This also implies that when the dissipative forces d_{ij} and N_i are zero, the dissipative part of stress is also zero. By Eq. (22) the $\lambda^{(a)}_i$ are not all independent but satisfy the relations

$$\begin{aligned}
\alpha_{10} - \alpha_{12} + \lambda^{(b)}_3 - \lambda^{(a)}_1 &= 0, \\
\alpha_{11} - \alpha_{13} + \lambda^{(b)}_4 - \lambda^{(b)}_2 - \lambda^{(a)}_4 - \lambda^{(a)}_2 &= 0, \\
\alpha_{14} - \alpha_{16} + \lambda^{(c)}_3 - \lambda^{(b)}_1 &= 0, \\
\alpha_{15} - \alpha_{17} + \lambda^{(c)}_4 - \lambda^{(c)}_2 - \lambda^{(b)}_4 - \lambda^{(b)}_2 &= 0, \\
\alpha_{18} - \alpha_{20} + \lambda^{(a)}_3 - \lambda^{(c)}_1 &= 0, \\
\alpha_{19} - \alpha_{21} + \lambda^{(a)}_4 - \lambda^{(a)}_2 - \lambda^{(c)}_2 - \lambda^{(c)}_4 &= 0
\end{aligned} \tag{31}$$

Thus combinations of the $\lambda_i^{(a)}$ determine the viscous torque and these combinations are all determined by different combinations of α_i so that the viscous torque can be expressed in terms of the α_i alone. As can be seen Eq. (30) shows that σ'_{ji} is determined by 21 constants while in the theory developed by Saupe[2] it is determined by 15 viscosity coefficients. The reason for this difference becomes clear once the dissipative function approach is adopted (Section 5).

Eq. (30) can also be derived by expressing σ'_{ji} as a linear sum of tensor products formed from d_{ij}, a_k, N_l^a (CP) and by using Eqs. (25) and (29). One finds 54 terms of the form $V_i W_j (U_k d_{kl} K_l)$ and 27 terms of the form $V_i W_j (N_k X_k)$ where U, V, W, X are chosen from (a, b, c). Orthorhombic symmetry is satisfied by 15 terms from the first set and by 6 terms from the second set. A linear superposition again results in σ'_{ji} of Eq. (30).

5. APPLICATION OF ONSAGER RECIPROCITY RELATIONS (ORR) AND THE DISSIPATIVE FUNCTION APPROACH

Using Eqs. (29), (30) and (31) the rate of entropy generation can be expressed as

$$\rho T\dot{S} = 2\varphi = \sigma_{kl}^s d_{kl} + \sigma_{ki}^a \Omega_{kl}^T \tag{32}$$

where σ_{kl}^s and σ_{kl}^a are the symmetric and antisymmetric parts of σ'_{kl} and φ the dissipative function. Following the traditional path[15,16] we treat d_{kl} and Ω_{kl}^T as forces and σ_{kl}^s and σ_{kl}^a as fluxes such that

$$\sigma_{ji}^s = D_{jirs}^{(1)} d_{rs} + D_{jirs}^{(2)} \Omega_{rs}^T, \qquad \sigma_{ji}^a = D_{jirs}^{(3)} d_{rs} + D_{jirs}^{(4)} \Omega_{rs}^T$$

whence by ORR

$$D_{jirs}^{(2)} = D_{rsji}^{(3)} \tag{33}$$

Using Eq. (33) one finds that

$$D_{jirs}^{(2)} = L_{jirs}^{ab}(\alpha_{10} + \alpha_{12}) + L_{jirs}^{bc}(\alpha_{14} + \alpha_{16}) + L_{jirs}^{ca}(\alpha_{18} + \alpha_{20})$$

$$D_{jirs}^{(3)} = L_{srji}^{ab}(\alpha_{11} - \alpha_{13}) + L_{srji}^{bc}(\alpha_{15} - \alpha_{17}) + L_{srji}^{ca}(\alpha_{19} - \alpha_{21}) \tag{34}$$

Eqs. (33) and (34) result in three Onsager relations

$$\alpha_{10} + \alpha_{12} = \alpha_{13} - \alpha_{11}; \qquad \alpha_{14} + \alpha_{16} = \alpha_{17} - \alpha_{15};$$
$$\alpha_{18} + \alpha_{20} = \alpha_{21} - \alpha_{19} \tag{35}$$

This reduces the number of independent α_i from 21 to 18. Thus mere application of ORR does not bring about formal accord between σ'_{ji} given by Eq. (30) and that given in ref. 2. In order to achieve this we start with the expression for entropy generation. With Eqs. (30), (31), (29) and (21)

$$\begin{aligned}
2\varphi = {} & \beta_1 A_a^2 + \beta_2 A_a B_b + \beta_3 B_b^2 + \beta_4 B_b C_c + \beta_5 C_c^2 \\
& + \beta_6 A_a C_c + \beta_7 N_k c_k A_b \\
& + \beta_8 A_b^2 + \beta_9 N_k a_k B_c + \beta_{10} B_c^2 + \beta_{11} N_k b_k A_c + \beta_{12} A_c^2 \\
& + \beta_{13} (N_k a_k)^2 + \beta_{14} (N_k b_k)^2 + \beta_{15} (N_k c_k)^2
\end{aligned} \tag{36}$$

where, using the definitions of η's from ref. 2,

$$\begin{aligned}
&\beta_1 = \alpha_1 = \eta_{aaaa}; \qquad \beta_2 = \alpha_1 + \alpha_2 + \alpha_4 + \alpha_6 = 2\eta_{aabb}; \\
&\beta_3 = \alpha_2 = \eta_{bbbb}; \qquad \beta_4 = \alpha_2 + \alpha_3 + \alpha_7 + \alpha_9 = 2\eta_{bbcc}; \\
&\beta_5 = \alpha_3 = \eta_{cccc}; \qquad \beta_6 = \alpha_1 + \alpha_3 + \alpha_5 + \alpha_8 = 2\eta_{ccaa}; \\
&\beta_7 = \alpha_{10} - \alpha_{11} + \alpha_{12} + \alpha_{13} = 2\gamma_{cab}; \qquad \beta_8 = \alpha_{11} + \alpha_{13} = 4\eta_{abab}; \\
&\beta_9 = \alpha_{14} - \alpha_{15} + \alpha_{16} + \alpha_{17} = 2\gamma_{abc}; \qquad \beta_{10} = \alpha_{15} + \alpha_{17} = 4\eta_{bcbc}; \\
&\beta_{11} = \alpha_{18} - \alpha_{19} + \alpha_{20} + \alpha_{21} = 2\gamma_{bca}; \qquad \beta_{12} = \alpha_{19} + \alpha_{21} = 4\eta_{caca}; \\
&\beta_{13} = \alpha_{16} - \alpha_{14} = \gamma_{aa}; \qquad \beta_{14} = \alpha_{20} - \alpha_{18} = \gamma_{bb}; \\
&\beta_{15} = \alpha_{12} - \alpha_{10} = \gamma_{cc};
\end{aligned} \tag{37}$$

As can be seen the 15 η's of ref. (2) can be written as combinations of α_i. However, all α_i cannot be calculated from Eq. (37). In order to do this we proceed as follows:

We now calculate the dissipative stress[2] σ_{ji}^D from

$$\sigma_{ji}^D = \sigma_{ji}^{Ds} + \sigma_{ji}^{Da} = \partial\varphi/\partial d_{ij} + \partial\varphi/\partial\omega_{ij} \tag{38}$$

and demand that

$$\sigma_{ji}^D \equiv \sigma'_{ji} \tag{39}$$

where σ'_{ji} is given by Eq. (30). [In Appendix II we show that this approach leads to the Parodi relation for uniaxial nematics. In addition we derive another equation which is the uniaxial analogue of Eq. (40) which is derived below. As mentioned in ref. 24, the stress σ_{ji} is always indeterminate to an additive term ζ_{ij} such that $\zeta_{ij,j} = 0$. This may mean that stress is actually never fully determined and thus Eq. (39) may not be strictly correct. However, the intention of this exercise is to show that if one adopts a procedure which is detailed in Eqs. (38) and (39), one can find the necessary additional equations which enable a complete formal comparison between the two approaches. It may be mentioned here that $\sigma_{ji}^{Da} = \partial\varphi/\partial\omega_{ij} \equiv -\partial\varphi/\partial\Omega_{ij}^{T}$. This definition is useful for constructing the stress tensor for dissipative relaxation of the director field involving no material flow.]

Eqs. (38) and (39) result in 21 equations of which only six are independent.

$$\alpha_2 + \alpha_4 = \alpha_1 + \alpha_6, \qquad \alpha_1 + \alpha_8 = \alpha_3 + \alpha_5,$$

$$\alpha_2 + \alpha_9 = \alpha_3 + \alpha_7 \tag{40}$$

$$\alpha_{10} + \alpha_{12} = \alpha_{13} - \alpha_{11}, \qquad \alpha_{14} + \alpha_{16} = \alpha_{17} - \alpha_{15},$$

$$\alpha_{18} + \alpha_{20} = \alpha_{21} - \alpha_{19} \tag{41}$$

Eq. (41) is ORR Eq. (35) which is recovered once more from the dissipative function approach. But Eq. (40) represents a different set of relationships. They correspond to additional equations which have to be satisfied by the viscosity coefficients if they are to form tensors; and also if these tensors should satisfy symmetry requirements. As is clear, the 15 β_i of Eq. (37) correspond to the 15 viscosity coefficients in Saupe's theory. Saupe[2] starts with an expansion of φ in terms of the dissipative forces d_{ij} and N_k and imposes orthorhombic symmetry on the viscosity tensors which connect the dissipative function with the dissipative forces. He obtains 15 independent quantities. However, in the present work, using the transformations (25), one gets an expression for σ_{ji}' which contains six more constants than σ_{ji}' of Saupe's theory. When the dissipative function approach is imposed on σ_{ji}' it results in ORR (which is in built) and additional relations (40) which force symmetry requirements on viscosity. Eq. (40) is independent of ORR because it relates viscosities which determine dependence of $\sigma_{ji}^{s\prime}$ on d_{ij}.

That there are 15 constants in Eq. (36) for φ, can be understood by the following considerations. Noting that φ is a quadratic form constructed out of a_i, b_j, c_k, d_{lm} and N_n we can construct φ from the products of the nine scalar quantities A_a, B_b, C_c, A_b, B_c, A_c (corresponding to the six independent components of d_{ij}) and $N_k a_k$, $N_k b_k$, $N_k c_k$ (corresponding to the three independent components of N_i). Out of the 45 products so obtained, only the 15 which occur in Eq. (36) are found to be admissible under orthorhombic symmetry.

With Eqs. (40) and (41) one can write the following relations between α_i, β_i and the η's of Saupe's theory:

$$\begin{aligned}
&\alpha_1 = \beta_1 = \eta_{aaaa}; \quad \alpha_2 = \beta_3 = \eta_{bbbb}; \quad \alpha_3 = \beta_5 = \eta_{cccc};\\
&\alpha_2 + \alpha_4 = \alpha_1 + \alpha_6 = \beta_2/2 = \eta_{aabb}; \quad \alpha_{11} + \alpha_{13} = \beta_8 = 4\eta_{abab};\\
&\alpha_3 + \alpha_7 = \alpha_2 + \alpha_9 = \beta_4/2 = \eta_{bbcc}; \quad \alpha_{15} + \alpha_{17} = \beta_{10} = 4\eta_{bcbc};\\
&\alpha_3 + \alpha_5 = \alpha_1 + \alpha_8 = \beta_6/2 = \eta_{ccaa}; \quad \alpha_{19} + \alpha_{21} = \beta_{12} = 4\eta_{caca};\\
&\alpha_{14} + \alpha_{16} = \alpha_{17} - \alpha_{15} = \beta_9/2 = \gamma_{abc}; \quad \alpha_{16} - \alpha_{14} = \beta_{13} = \gamma_{aa};\\
&\alpha_{18} + \alpha_{20} = \alpha_{21} - \alpha_{19} = \beta_{11}/2 = \gamma_{bca}; \quad \alpha_{20} - \alpha_{18} = \beta_{14} = \gamma_{bb};\\
&\alpha_{10} + \alpha_{12} = \alpha_{13} - \alpha_{11} = \beta_7/2 = \gamma_{cab}; \quad \alpha_{12} - \alpha_{10} = \beta_{15} = \gamma_{cc}
\end{aligned} \tag{42}$$

With Eq. (42), Eq. (30) can be recast into the form proposed by Saupe[2]

$$\begin{aligned}
\sigma'_{ji} = \frac{1}{2}\sum_a \Big[&\eta_{aaaa} A_a \delta_{ij} + a_i a_j \{ \eta_{aaaa} d_{kk} + B_b(2\eta_{aabb} - \eta_{aaaa} - \eta_{bbbb}) \\
&+ C_c(2\eta_{ccaa} - \eta_{cccc} - \eta_{aaaa}) \} \\
&+ a_i b_j \{ (4\eta_{abab} + \gamma_{cab}) A_b + N_l c_l (\gamma_{cab} + \gamma_{cc}) \} \\
&+ a_j b_i \{ (4\eta_{abab} - \gamma_{cab}) A_b + N_l c_l (\gamma_{cab} - \gamma_{cc}) \} \Big]
\end{aligned} \tag{43}$$

As can be seen, Eq. (43) satisfies transformations (25). However, if we construct a stress tensor which satisfies transformations (25), we get a different expression for stress. Taking a clue from Eq. (42), we can recast Eq. (30) into the form

$$\begin{aligned}
\sigma'_{ji} = \frac{1}{2}\sum_a \Big[&\eta_{aaaa} A_a \delta_{ij} + a_i a_j \{ \eta_{aaaa} d_{kk} + B_b(2\eta'_{aabb} - \eta_{aaaa} - \eta_{bbbb}) \\
&+ C_c(2\eta_{ccaa} - \eta_{aaaa} - \eta_{cccc}) \} \\
&+ a_i b_j \{ (4\eta_{abab} + \gamma_{cab}) A_b + (\gamma'_{cab} + \gamma_{cc}) N_l c_l \} \\
&+ a_j b_i \{ (4\eta_{abab} - \gamma_{cab}) A_b + (\gamma'_{cab} - \gamma_{cc}) N_l c_l \} \Big];
\end{aligned}$$

$$\begin{aligned}
&\alpha_2 + \alpha_4 = \eta'_{aabb}, \quad \alpha_3 + \alpha_7 = \eta'_{bbcc},\\
&\alpha_1 + \alpha_8 = \eta'_{ccaa}, \quad \alpha_1 + \alpha_6 = \eta_{aabb}\\
&\alpha_2 + \alpha_9 = \eta_{bbcc}, \quad \alpha_3 + \alpha_5 = \eta_{ccaa},\\
&\alpha_{13} - \alpha_{11} = \gamma_{cab}, \quad \alpha_{17} - \alpha_{15} = \gamma_{abc}\\
&\alpha_{21} - \alpha_{19} = \gamma_{bca}, \quad \alpha_{10} + \alpha_{12} = \gamma'_{cab},\\
&\alpha_{14} + \alpha_{16} = \gamma'_{abc}, \quad \alpha_{18} + \alpha_{20} = \gamma'_{bca}
\end{aligned} \tag{44}$$

The rest of the definitions are as in Eq. (42). Transition to Saupe's theory is made by equating the primed quantities to the corresponding unprimed ones. The above form of σ'_{ji} has been given only for writing it in a compact form; no assertion is made that the different viscosity coefficients form tensors. Though γ_{abc} and γ'_{abc} both determine entropy generation, only γ_{abc} enters into the expression for viscous torque, whose expression is formally identical to that given in ref. 2.

Positive definiteness of entropy generation requires that the viscosity coefficients satisfy the following inequalities

$$\sum_a \Big[\eta_{aaaa}(\eta_{bbcc} + \eta'_{bbcc})^2 - \frac{1}{3}\{4\eta_{aaaa}\eta_{bbbb}\eta_{cccc}$$
$$+ (\eta_{aabb} + \eta'_{aabb})(\eta_{bbcc} + \eta'_{bbcc})(\eta_{ccaa} + \eta'_{ccaa})\}\Big] \geq 0;$$
$$\eta_{aaaa} \geq 0; \qquad \gamma_{aa} \geq 0; \qquad \eta_{abab} \geq 0;$$
$$4\eta_{abab}\gamma_{cc} - (\gamma_{cab} + \gamma'_{cab})^2/4 \geq 0 \text{ (CP)} \tag{45}$$

The inequalities for Eq. (43) can be written by equating primed and unprimed quantities.

6. σ'_{ji} FOR A UNIAXIAL NEMATIC

Saupe[2] has shown that φ for a biaxial nematic reduces to that of a uniaxial nematic when uniaxial symmetry is imposed about one director field, say **a**. (Mention of a similar change for F has already been made in section 4.) For φ given by Eq. (36) Saupe[2] obtains the following relations:

$$\eta_{bbbb} = \eta_{cccc}; \qquad \eta_{aabb} = \eta_{ccaa};$$
$$\eta_{abab} = \eta_{caca}; \qquad \gamma_{bb} = \gamma_{cc}; \qquad \gamma_{bca} = -\gamma_{cab};$$
$$\eta_{bbcc} - \eta_{bbbb} = -2\eta_{bcbc}; \qquad \gamma_{aa} = 0 = \gamma_{abc} \tag{46}$$
$$\rho_1 = \eta_{bbcc}; \qquad \rho_2 = \rho_3 = \eta_{aabb} - \eta_{bbcc}; \qquad \mu_4 = 2\eta_{bcbc};$$
$$\mu_1 = \eta_{aaaa} + \eta_{bbbb} - 2\eta_{aabb} - 4\eta_{abab}; \qquad \mu_2 = -(\gamma_{bb} + \gamma_{bca})/2;$$
$$\mu_3 = (\gamma_{bb} - \gamma_{bca})/2; \qquad \mu_5 = (\gamma_{bca} - 2\eta_{bcbc})/2;$$
$$\mu_6 = -(\gamma_{bca} + 2\eta_{bcbc})/2 \tag{47}$$

The Parodi relation is naturally satisfied. The only difference from general forms of σ'_{ji} for uniaxial nematics[11,23,24] is that out of the three viscosities ρ_1, ρ_2, ρ_3 the last two are equal. This seems to agree with the compressible part of σ'_{ji} given by Forster *et al.*[25]

We find that it is possible to obtain Eqs. (46) and (47) from Eq. (43) without resorting to the dissipative function but by assuming uniaxial symmetry about **a** and by demanding that σ'_{ji} remain unaltered by arbitrary rotations about **a**. By demanding that **b** and **c** are equivalent and that σ'_{ji} should be independent of **b** and **c** one can deduce Eqs. (46) and (47).

The same procedure is adopted for σ'_{ji} given by Eq. (44) and the following relations are obtained

$$\eta_{bbbb} = \eta_{cccc}; \quad \eta'_{aabb} = \eta_{ccaa}; \quad \eta_{abab} = \eta_{caca};$$
$$\gamma_{bb} = \gamma_{cc}; \quad \gamma'_{cab} = -\gamma'_{bca}$$
$$\gamma_{cab} = -\gamma_{bca}; \quad \eta_{bbcc} = \eta'_{bbcc}; \quad \eta'_{ccaa} = \eta_{aabb};$$
$$\gamma_{aa} = 0 = \gamma_{abc} = \gamma'_{abc}; \quad \eta_{bbcc} - \eta_{bbbb} = -2\eta_{bcbc};$$
$$\rho_1 = \eta_{bbcc}; \quad \rho_2 = \eta_{ccaa} - \eta_{bbcc};$$
$$\rho_3 = \eta_{aabb} - \eta_{bbcc}; \quad \mu_1 = \eta_{aaaa} + \eta_{bbbb} - \eta_{aabb} - \eta'_{aabb} - 4\eta_{abab};$$
$$\mu_2 = -(\gamma_{bb} + \gamma'_{bca})/2; \quad \mu_3 = (\gamma_{bb} - \gamma'_{bca})/2; \quad \mu_4 = 2\eta_{bcbc};$$
$$\mu_5 = (\gamma_{bca} - 2\eta_{bcbc})/2; \quad \mu_6 = -(\gamma_{bca} + 2\eta_{bcbc})/2 \quad (48)$$

The differences between Eqs. (47) and (48) are (i) ρ_2 and ρ_3 are not equal in general. They become equal only if an additional condition $\eta'_{aabb} = \eta_{aabb}$ is imposed. If this is done, the expressions for μ_1 in Eqs. (47) and (48) will also become identical. (ii) ORR or the Parodi relation is not satisfied unless $\gamma'_{bca} = \gamma_{bca}$. Thus σ'_{ji} given by Eq. (44) reduces to a general expression for σ'_{ji} for uniaxial nematics.[11,23,24] Before concluding this section it must be mentioned that an interchange of the definitions of primed and unprimed quantities given in Eq. (44) leaves σ'_{ji} formally unaltered; the present definition ensures that the viscous torque in the present picture is described by an expression which is formally identical to that given in ref. 2.

7. SOME SIMPLE FLOWS

We consider simple flows whose apparent viscosity can help determine some combinations of viscosity coefficients. We start with simple shear flow[2,13] between plates $z = \pm h$ with constant director alignment $\mathbf{a} = (a_x, a_y, a_z)$ (CP) at all points of the sample. This may be possible at low shear rates with magnetic field and wall alignment. For a general velocity profile $\mathbf{v} = (v_x(z), v_y(z), 0)$, Eqs. (7) and (44) become

$$\sigma'_{zx} = J_1 v_{x,z} + J_2 v_{y,z} = r_1 = \text{constant};$$

$$\sigma'_{zy} = J_3 v_{x,z} + J_4 v_{y,z} = r_2 = \text{constant}; \qquad p_{,z} = \left[J_5 v_{x,z} + J_6 v_{y,z}\right]_{,z}$$

$$J_1 = \sum_a \left[a_x a_z b_x b_z (F_{ab} + G_{ab}) + a_z^2 b_x^2 D_{ab} + a_x^2 b_z^2 E_{ab}\right]$$

$$J_2 = \sum_a \left[a_x a_z b_y b_z F_{ab} + a_y a_z b_x b_z G_{ab} + b_x b_y a_z^2 D_{ab} + a_x a_y b_z^2 E_{ab}\right]$$

$$J_3 = \sum_a \left[a_y a_z b_x b_z F_{ab} + a_x a_z b_y b_z G_{ab} + b_x b_y a_z^2 D_{ab} + a_x a_y b_z^2 E_{ab}\right]$$

$$J_4 = \sum_a \left[a_y a_z b_y b_z (F_{ab} + G_{ab}) + a_y^2 b_z^2 E_{ab} + a_z^2 b_y^2 D_{ab}\right]$$

$$J_5 = \sum_a \left[a_x a_z C_{ab} + b_x b_z H_{ab}\right]; \qquad J_6 = \sum_a \left[a_y a_z C_{ab} + b_y b_z H_{ab}\right]$$

$$2C_{ab} = \eta_{aaaa} + b_z^2(2\eta_{aabb} - \eta_{bbbb} - \eta_{aaaa} + 4\eta_{abab})$$

$$4D_{ab} = 4\eta_{abab} + \gamma_{cc} - \gamma_{cab} - \gamma'_{cab};$$

$$4E_{ab} = 4\eta_{abab} + \gamma_{cc} + \gamma_{cab} + \gamma'_{cab}$$

$$2F_{ab} = 2\eta'_{aabb} - \eta_{aaaa} - \eta_{bbbb} + 2\eta_{abab} - \gamma_{cc}/2 + (\gamma_{cab} - \gamma'_{cab})/2$$

$$2G_{ab} = 2\eta_{aabb} - \eta_{aaaa} - \eta_{bbbb} + 2\eta_{abab} - \gamma_{cc}/2 + (\gamma_{cab} - \gamma'_{cab})/2$$

$$2H_{ab} = a_z^2[2\eta'_{aabb} - \eta_{aaaa} - \eta_{bbbb} + 4\eta_{abab}] \text{ (CP)} \tag{49}$$

For plates $z = \pm h$ which are sliding along $\pm x$ with velocities $\pm V/2$, no slip at the boundaries requires that $v_x(\pm h) = \pm V/2$, $v_y(\pm h) = 0$. From Eq. (49) one immediately finds that the shear rate $v_{x,z} = V/2h$ = constant, $v_y = 0$ and p = constant. But there is a net transverse shear stress

$$\sigma'_{zy} = r_2 = r_1 J_3/J_1 \tag{50}$$

The apparent viscosity is given by

$$\eta = \sigma'_{zx}/v_{x,z} = J_1 \tag{51}$$

J_1 reduces to the expression which is given by Saupe.[2] Measurement of J_1 in different configurations should, in principle yield values of

$$2(\eta_{aabb} + \eta'_{aabb} - \eta_{aaaa} - \eta_{bbbb}) + (4\eta_{abab} - \gamma_{cc}),$$
$$4\eta_{abab} + \gamma_{cc}, \gamma_{cab} + \gamma'_{cab} \text{ (CP)} \tag{52}$$

Plane Poiseuille flow is more interesting. On the basis of earlier experiments on uniaxial nematics demonstrating transverse flow and pressure effects[26] we integrate Eq. (7) to get

$$J_1 v_{x,z} + J_2 v_{y,z} = p_{,x} z; \qquad J_3 v_{x,z} + J_4 v_{y,z} = p_{,y} z \tag{53}$$

where $v_x(\pm h) = 0 = v_y(\pm h)$ and $p_{,x}$ = constant, $p_{,y}$ = constant are the pressure gradients along x and y respectively. Eq. (53) shows that a pressure difference along x can generate one along y and vice versa, for general director orientation. Eq. (53) along with the boundary conditions on the two velocity components results in

$$v_x = (J_4 p_{,x} - J_2 p_{,y})(z^2 - h^2)/2\Delta$$
$$v_y = (J_1 p_{,y} - J_3 p_{,x})(z^2 - h^2)/2\Delta; \qquad \Delta = J_1 J_4 - J_2 J_3 \tag{54}$$

For wide enough rectangular capillaries, it may be possible to observe diversion of flow lines towards y when $p_{,x}$ is impressed on the fluid. However if transverse flow v_y is frustrated in a narrow capillary, the transverse pressure difference per unit width

$$p_{,y} = J_3 p_{,x}/J_1 \tag{55}$$

Apparent viscosity is again J_1. It can be checked from Eqs. (49) and (55) that $p_{,y}$ vanishes when the director orientation is symmetrical relative to the flow direction. [For example, $\mathbf{a} = (1,0,0), \mathbf{b} = (0,1,0)$]. This also holds for Eq. (50) which describes the transverse shear stress in simple shear flow. Determination of $p_{,y}$ for general orientations may help a separate determination of F_{ab} and G_{ab} (CP). According to the dissipative function approach, $F_{ab} = G_{ab}$ (CP). Thus a study of plane Poiseuille flow may help in a formal comparison of the dissipative function approach with the approach which is adopted in this work.

General flow alignment of a biaxial nematic under simple shear is a complex problem. One can however consider special cases as done by Saupe[2] where one director vector is normal to the shear plane and the other two directors are in the shear plane. It is possible to find out the condition under which flow can align the two directors in the flow plane; one can also study the stability of flow alignment. Since the viscous torque expression is identical in the two pictures, one gets the same conditions for flow alignment. However since the total viscous stress is different in the two pictures one may get a different condition for stability of the flow alignment in the present picture. A complete treatment of stability would involve inclusion of elastic torques. This as well as the roles played by the different viscosities in determining the viscous torque will be considered separately. Shear and plane Poiseuille flows help determine only certain combinations of viscosities. It may be necessary to study other situations such as back flow under director field relaxation, effect of a rotating magnetic field, propagation of compressional waves etc. to get a more complete understanding of the other viscosity coefficients.

Improvements of the present approach will involve considering change of director magnitude and the effect of temperature gradients.[13] It also seems interesting to study the Saupe and the Ericksen–Leslie approaches for the other two classes of biaxial symmetries—monoclinic and triclinic.[21]

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APPENDIX I

Velocity of Rotation N of the director field a, b, c relative to the fluid

Since N_i^a is perpendicular to a_k, $N_i^a a_i = 0$ or

$$N_i^a = \Omega_{ik}^T a_k; \qquad \text{with } \Omega_{ik}^T = -\Omega_{ki}^T \tag{A1}$$

where Ω_{ik}^T represents rotation of a_j relative to the fluid. Since a_i, b_j, c_k are orthonormal, Ω_{ik}^T should represent the rotations of b_i and c_k relative to the fluid. Hence

$$N_i^b = \Omega_{ik}^T b_k; \qquad N_i^c = \Omega_{ik}^T c_k \tag{A2}$$

Since $N_i^a a_i = 0$, N_i^a lies in the $(\mathbf{b}, \mathbf{c})$ plane so that

$$R_a^a = N_i^a a_i = 0; \qquad R_b^a = N_i^a b_i; \qquad R_c^a = N_i^a c_i \text{ (CP)} \tag{A3}$$

Since $a_i b_i = 0$ (CP),

$$R_b^a = -R_a^b \text{ (CP)} \tag{A4}$$

In the principal axes frame (a_i, b_j, c_k), Eqs. (A1) can be written as

$$N_\xi^a = \Omega_{\xi\lambda}^T a_\lambda = \Omega_{\xi a}^T; \qquad \Omega_{\xi\lambda}^T = R_\xi^\lambda(\lambda, \xi = a, b, c) \tag{A5}$$

The axial vector N_i which represents the rotational velocity of the director field relative to the fluid is defined by

$$N_\xi = \tfrac{1}{2} e_{\xi\alpha\beta} R_\beta^\alpha = \tfrac{1}{2} e_{\xi\alpha\beta} \Omega_{\beta\alpha}^T \tag{A6}$$

From (A3), (A4) and (A6) one can write in frame indifferent notation

$$\mathbf{N} = \sum_a (\mathbf{N}^b \cdot \mathbf{c})\mathbf{a} = \Omega_{kl}^T \sum_a b_l c_k a_i \tag{A7}$$

APPENDIX II

The Dissipative Function Approach for Uniaxial Nematics

Following ref. 15, the entropy generation per unit volume per unit time for a compressible nematic liquid crystal under flow is given by

$$\rho T\dot{S} = t'_{ji} d_{ij} - g'_i N_i = 2\varphi \tag{B1}$$

where

$$\begin{aligned}
t'_{ji} &= \left[\rho_1 \delta_{ij} d_{kk} + \rho_2 n_i n_j d_{kk}\right] + \rho_3 \delta_{ij}(d_{kl} n_k n_l) \\
&\quad + \mu_1 (d_{km} n_k n_m) n_i n_j + \mu_2 n_j N_i + \mu_3 n_i N_j + \mu_4 d_{ij} \\
&\quad + \mu_5 n_j d_{ik} n_k + \mu_6 n_i d_{jk} n_k \quad \text{(ref. 30.)} \\
N_i &= \dot{n}_i - \omega_{ik} n_k, \qquad g'_i = \lambda_1 N_i + \lambda_2 d_{ij} n_j, \\
\lambda_1 &= \mu_2 - \mu_3, \qquad \lambda_2 = \mu_5 - \mu_6
\end{aligned} \tag{B2}$$

Using Eq. (B2), Eq. (B1) can be written as

$$\begin{aligned}
2\varphi &= \rho_1 d_{kk} d_{ll} + (\rho_2 + \rho_3) d_{ll} d_{kp} n_k n_p + \mu_1 (d_{kp} n_k n_p)^2 \\
&\quad + \mu_4 d_{pq} d_{pq} + (\mu_2 + \mu_3 - \mu_5 + \mu_6) d_{pq} n_p N_q \\
&\quad + (\mu_5 + \mu_6)(d_{pq} n_q d_{pr} n_r) - \lambda_1 N_q N_q
\end{aligned} \tag{B3}$$

Following ref. 2, we define the dissipative stress as

$$t_{ji}^D = t_{ji}^{Ds} + t_{ji}^{Da} \equiv \partial\varphi/\partial d_{ij} + \partial\varphi/\partial\omega_{ij} \tag{B4}$$

Eqs. (B3) and (B4) reduce to

$$
\begin{aligned}
t_{ji}^{D} = {} & \rho_1 d_{kk}\delta_{ij} + (\rho_2 + \rho_3)\delta_{ij} d_{kp} n_k n_p/2 + (\rho_2 + \rho_3) d_{ll} n_i n_j/2 \\
& + \mu_1 d_{kp} n_k n_p n_i n_j + \mu_4 d_{ij} + n_j N_i (3\mu_2 - \mu_3 - \mu_5 + \mu_6)/4 \\
& + n_i N_j (3\mu_3 - \mu_2 - \mu_5 + \mu_6)/4 \\
& + n_j d_{ik} n_k (3\mu_5 + \mu_6 - \mu_2 - \mu_3)/4 \\
& + n_i d_{jk} n_k (3\mu_6 + \mu_5 + \mu_2 + \mu_3)/4
\end{aligned} \tag{B5}
$$

If we demand that $t'_{ji} \equiv t_{ji}^{D}$ we get two equations:

$$\mu_2 + \mu_3 = \mu_6 - \mu_5 \tag{B6}$$

which is Parodi's relation and

$$\rho_2 = \rho_3 \tag{B7}$$

which is similar to Eq. (40) for a biaxial nematic. [Formally identical results are obtained by using the definition $t_{ji}^{Da} = -\partial\varphi/\partial\Omega_{ij}^{T}$, where $N_i = \Omega_{ij}^{T} n_j$]

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